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$\mathbb{R}$-trees and laminations for free groups II: The dual lamination of an $\mathbb{R}$-tree

Thierry Coulbois, Arnaud Hilion and Martin Lustig

June 9, 2007

1 Introduction

A geodesic lamination $\mathfrak{L}$ on a hyperbolic surface S, provided with a transverse measure, defines (via the lift $\widetilde{\mathfrak{L}}$ of $\mathfrak{L}$ to the universal covering of S) an action of $\pi_1 S$ on an $\mathbb{R}$-tree T which is often called *dual* to the lamination $\widetilde{\mathfrak{L}}$. Conversely, every small action of $\pi_1 S$ on an $\mathbb{R}$-tree T comes from this construction, provided the surface is closed and the action on T is *small* [Sko96].

A generalization of this concept occurs first in the work of E. Rips, and is since then widely used. A particular kind of $\mathbb{R}$-tree actions can be defined as spaces dual to measured foliations on finite 2-complexes. If the latter is not a surface (and can not be made into a surface by certain elementary moves), then the resulting $\mathbb{R}$-tree is qualitatively different from the ones dual to surface laminations, see [GLP94, BF95, LP97].

A third occurence of laminations in direct relation to $\mathbb{R}$-trees takes place in the context of free group automorphisms, specifically for *irreducible automorphisms with irreducible powers (= iwip automorphisms).* For such an $\alpha \in \mathrm{Aut}(F_N)$ every (non-periodic) conjugacy class of elements in F_N converges to a collection of biinfinite *legal* paths on a train track representing α, see [Lus92, BFH97]. On the other hand (compare [BF], [GJLL98]), the train track itself converges towards an $\mathbb{R}$-tree T_α which is projectively fixed under the induced action of α on the set of $\mathbb{R}$-tree actions of F_N. In the particular case where α is induced by a pseudo-Anosov homeomorphism φ of some surface S (with one boundary component), then the above collection of biinfinite legal paths is in precise 1 - 1 correspondence with the leaves of the stable lamination of φ.

Finally, in [LL03] a collection of one-sided infinite words reminiscent to half-leaves of a lamination was associated to an arbitrary very small F_N-action on an $\mathbb{R}$-tree T, as a tool to prove that iwip automorphisms of F_N have a North-South dynamics on the space $\overline{\mathrm{CV}}_N$ of very small $\mathbb{R}$-tree actions of F_N.

This puzzle of coinciding and consistent observations induced the authors to set out for a general theory, in the realm of free (and perhaps later word-hyperbolic) groups. As a first step, in [CHL-I] algebraic laminations were defined, generalizing at the same time geodesic laminations on surfaces, as well as symbolic flows as known from discrete dynamics. They come in three equivalent languages, group theoretic, dynamic and combinatorial, and passing from one to the other turns out to be rather helpful. In [CHL-I] these "translations" were established with some care, and the topology, the partial order, as well as the natural $\mathrm{Out}(F_N)$-action were studied.

In the present paper we use these tools to define, for any isometric F_N-action on an $\mathbb{R}$-tree T, a set

$$L^2(T) \subset \partial^2 F_N := \partial F_N \times \partial F_N \smallsetminus \Delta\,,$$

where ∂F_N denotes the Gromov boundary of the free group F_N, and Δ is the diagonal. The set $L^2(T)$ is empty if the F_N-action on T is free and discrete (simplicial), and it is an algebraic lamination otherwise. There are several competing natural approaches to define $L^2(T)$, which we present briefly below. Working out the precise relationship between them is the core content of this paper.

1. (see §4) The lamination $L^2_\Omega(T)$ is defined for all isometric actions of F_N on an $\mathbb{R}$-tree T: For every $\varepsilon > 0$ we consider the set $\Omega_\varepsilon(T)$ of all elements $g \in F_N$ with translation length on T that satisfies:

$$\|g\|_T \ < \varepsilon$$

The set $\Omega_\varepsilon(T)$ generates an algebraic lamination $L^2_\varepsilon(T)$ which is the smallest lamination that contains every $(g^{-1}g^{-1}g^{-1}\ldots, ggg\ldots) \in \partial^2 F_N$ with $g \in \Omega_\varepsilon(T)$. We define $L^2_\Omega(T)$ to be the intersection of all $L^2_\varepsilon(T)$.

We are most interested in $\mathbb{R}$-trees T where every F_N-orbit of points is dense in T. To any such T we associate in this paper two more laminations, which are of rather different nature:

2. (see §5) In order to define the lamination $L^2_\infty(T)$ one first fixes a basis $\mathcal{A}$ of F_N. Then one picks an arbitrary point $P \in T$ and considers the set $L^1_{\mathcal{A}}(T)$ of one-sided infinite reduced words $x_1x_2\ldots$ in the basis $\mathcal{A}$ such that the set of all $x_1x_2\ldots x_kP$ has bounded diameter in T. One immediately observes that the set $L^1_{\mathcal{A}}(T)$ is independent of the choice of P. As next step, one considers the language $\mathcal{L}_{\mathcal{A}}(L^1_{\mathcal{A}}(T))$ derived from $L^1_{\mathcal{A}}(T)$, i.e. the set of all finite subwords of any $x_1x_2\ldots \in L^1_{\mathcal{A}}(T)$, and its *recurrent* sublanguage $\mathcal{L}^\infty_{\mathcal{A}}(L^1_{\mathcal{A}}(T))$: The latter consists precisely of those words which occur infinitely often as subword in some $x_1x_2\ldots \in L^1_{\mathcal{A}}(T)$. The advantage of this language is that it is *laminary*, and thus it defines canonically an algebraic lamination $L^2_\infty(T)$ (see [CHL-I]). We prove in detail in §5 that this algebraic lamination does not depend on the basis $\mathcal{A}$ used in the construction sketched above.

Aside: The passage from $L^1(T)$ to the recurrent language $\mathcal{L}^\infty_{\mathcal{A}}(L^1_{\mathcal{A}}(T))$ is subtle but rather important: It turns out that, contrary to $L^2_\infty(T)$, the set $L^1(T)$ depends not just on the topology, but actually on the metric of T. This is discussed in detail in [CHL05] and [CHLL].

3. (see §8) If T is an $\mathbb{R}$-tree dual to a measured lamination $\mathfrak{L}$ on surface S, then a leaf l of the lift $\widetilde{\mathfrak{L}}$ of $\mathfrak{L}$ to the universal covering $\widetilde{S}$ determines on one hand a point x_l in the dual tree T, and on the other hand two limit points $P_+, P_- \in \partial\widetilde{S}$ on the boundary at infinity $\partial\widetilde{S}$ of $\widetilde{S}$:

$$\{P_+, P_-\} = \partial l\,.$$

Note that, in the case where S has non-empty boundary, $\partial\widetilde{S}$ is canonically identified with the Gromov boundary ∂F_N of $F_N = \pi_1 S$.

In [LL03] this correspondence has been generalized to an F_N-equivariant map $\mathcal{Q} : \partial F_N \to \overline{T} \cup \partial T$, for any $\mathbb{R}$-tree T with dense F_N-orbits, where $\overline{T}$ denotes the metric completion of T, and ∂T the Gromov boundary. The definition of the map $\mathcal{Q}$ is reviewed in §6, and the geometric meaning of $\mathcal{Q}$ is explained in more detail in §7. In the above special case one gets:

$$\mathcal{Q}(P_+) = \mathcal{Q}(P_-) = x_l \in T$$

This motivates the definition of the lamination $L^2_{\mathcal{Q}}(T)$, which consists of all pairs $(X, X') \in \partial^2 F_N$ that determine the same limit point $\mathcal{Q}(X) = \mathcal{Q}(X')$ in $\overline{T}$.

The main result of this paper, proved in two steps (Propositions 5.8 and 8.5), is:

Theorem 1.1. *For every very small $\mathbb{R}$-tree with F_N-action that has dense orbits the above described three algebraic laminations coincide:*

$$L^2_{\Omega}(T) = L^2_{\infty}(T) = L^2_{\mathcal{Q}}(T)$$

This defines a dual algebraic lamination $L^2(T)$ canonically associated to T.

If the F_N-action on T does not have dense orbits, we define $L^2(T) = L^2_{\Omega}(T)$. For any non-simplicial such T there is a canonical (maximal) quotient $\mathbb{R}$-tree T' which has dense orbits, and we show that $L^2(T) \subset L^2(T')$ (see Remark 2.1 and Remark 4.4).

In a subsequent third paper we go one step further and consider invariant measures μ (called *currents*) on $L^2(T)$. We show in [CHL-III] that such a current defines a *dual metric* d_μ on T: If T is dual to a surface lamination $\mathfrak{L}$ as in the beginning of this introduction, and if μ comes from a transverse measure on $\mathfrak{L}$, then d_μ is indeed the $\mathbb{R}$-tree metric on the dual tree T. In general, however, it is shown in [CHL-III] that this dual metric can have very exotic properties.

Note also that the lamination $L^2(T)$ introduced in this paper has been used successfully in [CHL05] to characterize the underlying topological structure of $\mathbb{R}$-trees which stays invariant when the metric is F_N-equivariantly changed (so called "non-uniquely ergometric $\mathbb{R}$-trees").

The dual lamination $L^2(T)$ plays also a crucial role in establishing a continuous and $\mathrm{Out}(F_N)$-equivariant map from a large part of the boundary of Outer space into the space of projectivized currents, see [CHL3] and [KL]. It is also the basis for work in progress of the third author with I. Kapovich on perpendicular ($\mathbb{R}$-tree, current)-pairs.

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2 F_N-actions on $\mathbb{R}$-trees

In this section we recall some of the known properties of actions of a free group F_N on an $\mathbb{R}$-tree T. For details and background see [Vog02, Sha87] and the references given there.

An $\mathbb{R}$-tree is a metric space which is 0-hyperbolic and geodesic. Alternatively, an $\mathbb{R}$-tree is a metric space (T, d) where any two points $P, Q \in T$ are joined by a unique arc and this arc is isometric to the interval $[0, d(P, Q)] \subset \mathbb{R}$.

In this paper an $\mathbb{R}$-tree T always comes with a left action of F_N on T by isometries. Any isometry w of T is either *elliptic*, in which case it fixes at least one point of T, or else it is *hyperbolic*, in which case there is an *axis* $\mathrm{Ax}(w)$ in T, isometric to $\mathbb{R}$, which is w-invariant, and along which w acts as translation. The *translation length*

$$\|w\|_T = \inf\{d(P, wP) \mid P \in T\}$$

agrees in the hyperbolic case with $d(Q, wQ)$ for any point Q in $\mathrm{Ax}(w)$, while in the elliptic case it is 0. The F_N-action on T is called *abelian* if there exists a homomorphism $\psi : F_N \to \mathbb{R}$ such that $\|w\|_T = |\psi(w)|$ for all $w \in F_N$. In this case there exists an infinite ray $\rho \subset T$ such that every $w \in F_N$ acts as translation for every point sufficiently far out on ρ. The $\mathbb{R}$-tree actions treated in this paper will all turn out to be non-abelian.

We always assume that T is a *minimal* $\mathbb{R}$-tree, i.e. there is no non-empty F_N-invariant proper subtree of T. Another minimal $\mathbb{R}$-tree T' with isometric F_N-action is F_N-equivariantly isometric to T if and only if one has

$$\|w\|_T = \|w\|_{T'}$$

for every element $w \in F_N$, and if the actions are non-abelian. The set of $\mathbb{R}$-trees equipped with such F_N-actions inherits a topology from its image in $\mathbb{R}^{F_N}$ under the map

$$T \mapsto (\|w\|_T)_{w \in F_N} \in \mathbb{R}^{F_N} .$$

A tree (or a tree action) is called *small* if any two group elements that fix pointwise a non-trivial arc in T do commute. It is called *very small* if moreover (i) the fixed set $Fix(g) \subset T$ of any elliptic element $1 \neq g \in F_N$ is a segment or a single point (i.e. "no branching"), and (ii) $Fix(g) = Fix(g^m)$ for all $g \in F_N$ and $m \geq 1$. One sees easily that every small (and thus every very small) action is non-abelian.

The particular case of *simplicial* $\mathbb{R}$-trees T with isometric F_N-actions, which have trivial edge stabilizers, arises from graphs Γ with a *marking isomorphism*

$F_N \cong \pi_1\Gamma$, where the edges of Γ are given a non-negative length, which is for at least one of them strictly positive: The simplicial $\mathbb{R}$-tree T is then given by lifting the edge lengths to the the universal covering $\widetilde{\Gamma}$, equipped with the action of F_N by deck transformations. (Note that there is a minor ambiguity concerning the the topology on the set $T = \widetilde{\Gamma}$: the metric topology on T is in general weaker than the cellular topology on $\widetilde{\Gamma}$!). If every edge length of Γ is non-zero, then the action of F_N on T is free. The space cv_N of $\mathbb{R}$-trees equipped with a free simplicial action has been introduced by M. Culler and K. Vogtmann. Its closure $\overline{\mathrm{cv}}_N$ (in the space $\mathbb{R}^{F_N}$ as described above) is precisely the set of all of the above mentioned very small $\mathbb{R}$-trees. The *boundary* $\overline{\mathrm{cv}}_N \smallsetminus \mathrm{cv}_N$ is denoted by $\partial\mathrm{cv}_N$. One often normalizes Γ to have volume 1, thus obtaining the subspace CV_N of cv_N, which has been named *Outer space* by P. Shalen. Alternatively, one can *projectivize* the space of tree actions: two trees T and T' are in the same equivalence class $[T]$ if they are F_N-equivariantly homothetic. This projectivization maps $\overline{\mathrm{cv}}_N$ onto a compact space $\overline{\mathrm{CV}}_N$, which contains a homeomorphic copy of CV_N, called the *interior*, and the projectivized image $\partial\mathrm{CV}_N$ of $\partial\mathrm{cv}_N$, called the *boundary*. Both CV_N and $\overline{\mathrm{CV}}_N$ are contractible and finite dimensional. For more information see [Vog02, CV86].

The group $\mathrm{Out}(F_N)$ acts by homeomorphisms on cv_N and $\partial\mathrm{cv}_N$, as well as on CV_N and $\partial\mathrm{CV}_N$, and the action on CV_N is properly discontinuous (though not free). These actions, specified in §9, provide valuable information about the group $\mathrm{Out}(F_N)$. Note also that there is a strong similarity between Outer space CV_N with the $\mathrm{Out}(F_N)$-action on one side, and Teichmüller space with the action of the mapping class group on the other. The only substantial difference is that CV_N is not a manifold.

In the second half of this paper we will concentrate on the particularly interesting case where some (and hence every) F_N-orbit of points is dense in T. That this "dense orbits" hypothesis is not very restrictive follows from the following consideration:

Remark 2.1. Every $\mathbb{R}$-tree $T \in \mathrm{cv}_N \cup \partial\mathrm{cv}_N$ decomposes canonically into two disjoint F_N-invariant (possibly empty) subsets T_d and T_c, where the former is given as the union of all points $P \in T$ such that the orbit $F_N P$ is a discrete (or, equivalently, a closed discrete) subset of T, and the latter is the complement $T \smallsetminus T_d$. Using property (ii) in the above definition of a very small action, the subset $T_c \subset T$ is easily seen to be closed, and each connected component of it is a subtree T' of T with the property that the subgroup U of F_N that stabilizes T' acts on T' with *dense orbits*: $T' = \overline{UP}$ for any $P \in T'$. Unless T_c is empty, in which case the set of branch points is a discrete subset of T and thus T is simplicial, we can contract the closure of every connected component of T_d to a single point, to get the canonical maximal non-trivial quotient tree T/T_d on which now all of F_N acts minimally and with dense orbits. Compare [L94].

3 Bounded Back Tracking

Every small action on an $\mathbb{R}$-tree is known to have the *bounded backtracking property (BBT)* (see [GJLL98]), which is of great use in this paper:

Let Γ be any (non-metric) graph with a marking isomorphism $\pi_1\Gamma \cong F_N$, and let $\widetilde{\Gamma}$ be its universal covering. Let $i : \widetilde{\Gamma} \to T$ be any F_N-equivariant map. Then the map i *satisfies BBT* if and only if for every pair of points $P, Q \in \widetilde{\Gamma}$ the i-image of the geodesic segment $[P, Q] \subset \widetilde{\Gamma}$ is contained in the C-neighborhood of $[i(P), i(Q)] \subset T$, where $C \geq 0$ is an a priori constant independent of the choice of P and Q. We denote by $\mathrm{BBT}(i) \geq 0$ the smallest such constant.

Every $\mathbb{R}$-tree T with isometric F_N-action admits a map i as above and, i satisfies BBT if and only if any other such map $i' : \widetilde{\Gamma}' \to T$ also satisfies BBT. Hence the property BBT is a well defined property of the tree T.

We can assume that the above map $i : \widetilde{\Gamma} \to T$ is *edge-geodesic*: i maps every edge $e \subset \widetilde{\Gamma}$ to the geodesic segment that connects the images of the endpoints of e. One can make Γ into a metric graph by giving each edge of Γ and each of its lifts e to $\widetilde{\Gamma}$ the length of $i(e)$. Without loss of generality one can assume that the metric on every edge e is properly distributed so that i is actually *edge-isometric*, i.e. i maps every edge of $\widetilde{\Gamma}$ isometrically onto its image. In this case the inequality

$$BBT(i) \leq \mathrm{vol}(\Gamma)$$

has been proved in [GJLL98], where the *volume* $\mathrm{vol}(\Gamma)$ of Γ is the sum of the lengths of its edges.

A particular choice of Γ, for any base $\mathcal{A}$ of F_N, is the rose $\mathcal{R}_\mathcal{A}$ with n leaves that are in one-to-one correspondence with the elements of $\mathcal{A}$. In this case the universal covering $\widetilde{\mathcal{R}}_\mathcal{A}$ is canonically identified with the Cayley graph of F_N with respect to the generating system $\mathcal{A}$, and the edge-geodesic map $i = i_{P,\mathcal{A}}$ is uniquely determined by the choice of a *base point* $P = i(\mathbf{1})$, where $\mathbf{1}$ is the vertex of the Cayley graph that corresponds to the neutral element $1 \in F_N$. In this case we denote the BBT-constant $\mathrm{BBT}(i_{P,\mathcal{A}})$ by $\mathrm{BBT}(\mathcal{A}, P)$, and the volume of $\mathcal{R}_\mathcal{A}$ by $\mathrm{vol}(\mathcal{A}, P)$.

Lemma 3.1. *Let T be an $\mathbb{R}$-tree with isometric F_N-action that satisfies BBT. Let $\mathcal{A}$ be any basis of $F_N = F(\mathcal{A})$, and let P be any point of T. Then the constants $BBT(\mathcal{A}, P) \geq 0$ and $vol(\mathcal{A}, P) > 0$ satisfy:*

(a) For any reduced word $w = x_1 x_2 \ldots x_n$ in $F(\mathcal{A})$ and any prefix $v = x_1 x_2 \ldots x_m$ of w the point vP is contained in the $BBT(\mathcal{A}, P)$-neighborhood of the geodesic segment $[P, wP] \subset T$.

(b) For any cyclically reduced word w in $F(\mathcal{A})$ one has:

$$d(wP, P) \leq 2BBT(\mathcal{A}, P) + \|w\|_T$$

(c) Any subword u of a cyclically reduced word $w \in F(\mathcal{A})$ satisfies:

$$d(uP, P) \leq 2BBT(\mathcal{A}, P) + \|w\|_T$$

(d) Every $x \in \mathcal{A} \cup \mathcal{A}^{-1}$ satisfies:

$$d(P, xP) \leq vol(\mathcal{A}, P)$$

Proof. (a) We only need to observe that a reduced word w in $\mathcal{A}$ defines a geodesic segment $[\mathbf{1}, w\mathbf{1}]$ in $\widetilde{\mathcal{R}}_{\mathcal{A}}$, and apply the definition of $\mathrm{BBT}(\mathcal{A}, P)$.

(b) As w is cyclically reduced, wP and w^2P are contained in the $\mathrm{BBT}(\mathcal{A}, P)$-neighborhood of the segment $[P, w^3P]$, and thus P and wP are contained in the $\mathrm{BBT}(\mathcal{A}, P)$-neighborhood of the segment $[w^{-1}P, w^2P]$. As the axis $\mathrm{Ax}(w)$ contains $[w^{-1}P, wP] \cap [P, w^2P]$, and the latter is the fundamental domain with respect to the action of w on $\mathrm{Ax}(w)$ and hence has length $\|w\|_T$, the desired inequality follows.

(c) As in (b) we see that P and uP are contained in the $\mathrm{BBT}(\mathcal{A}, P)$-neighborhood of the intersection $[w^{-1}P, wP] \cap [P, w^2P]$ and thus of $Ax(w)$.

(d) This is a direct consequence of the above definition of the volume. □

The following statement has been shown in [LL03], Remark 2.6:

Lemma 3.2. *Let T be an $\mathbb{R}$-tree with a very small action with dense orbits of F_N. For any point P in T, there exists a sequence of bases $\mathcal{A}_k$ such that the two sequences of constants $BBT(\mathcal{A}_k, P)$ and $vol(\mathcal{A}_k, P)$ both tend to 0, for $k \to \infty$.* □

4 The dual lamination associated to an $\mathbb{R}$-tree

In [CHL-I], §2, *algebraic laminations* have been defined as subsets of $\partial^2 F_N$ (defined in §1 above). In particular, for every $w \in F_N$ the *rational* algebraic lamination $L^2(w)$ has been introduced. In §§4 and 5 of [CHL-I] *laminary languages* over a basis $\mathcal{A}$ of F_N have been presented, and the bijection $\rho^{\mathcal{A}}_{\mathcal{L}} \rho^2_{\mathcal{A}}$ has been established which associates in a natural way to any algebraic lamination $L^2 \subset \partial^2 F_N$ a laminary language, denoted here by $\mathcal{L}_{\mathcal{A}}(L^2)$, compare Theorem 1.1 of [CHL-I]. We use the notation from the predecessor article [CHL-I] to facilitate the reading of this section, but we also simplify some of the notation for the convenience of the reader. For example, if $\mathcal{L} = \mathcal{L}_{\mathcal{A}}(L^2)$ is the laminary language canonically associated to the algebraic lamination L^2, we will write in the next section $L^2 = L^2(\mathcal{L})$.

Let T be an $\mathbb{R}$-tree with isometric F_N-action. For every $\varepsilon > 0$ we consider the set

$$\Omega_\varepsilon(T) \;=\; \{w \in F_N \mid \|w\|_T < \varepsilon\} \;\subset\; F_N$$

which is invariant under conjugation and inversion in F_N, and the set

$$\Omega^2_\varepsilon(T) \;=\; \bigcup_{w \in \Omega_\varepsilon(T)} L^2(w) \;\subset\; \partial^2 F_N$$

which is invariant under the action of F_N and of the flip-map on $\partial^2 F_N$. We note that either the F_N-action on T is free simplicial, i.e. T belongs to cv_N, or else $\Omega_\varepsilon(T)$ and hence $\Omega^2_\varepsilon(T)$ are non-empty for any $\varepsilon > 0$. In the latter case we pass to the closure in $\partial^2 F_N$ to obtain an algebraic lamination:

$$L^2_\varepsilon(T) \;=\; \overline{\Omega^2_\varepsilon(T)} \;\subset\; \partial^2 F_N\,.$$

By Lemma 4.2 of [CHL-I] we can define:

Definition 4.1. Let T be an $\mathbb{R}$-tree on which F_N acts by isometries. If T belongs to cv_N, then we define $L^2_\Omega(T)$ to be the empty set (which is not an algebraic lamination!). Otherwise we define the *dual algebraic lamination* associated to T as follows:

$$L^2_\Omega(T) = \bigcap_{\varepsilon>0} L^2_\varepsilon(T)\,.$$

We note that $L^2_\Omega(T)$ depends only on the projective class $[T] \in \overline{\mathrm{CV}}_N$.

For any basis $\mathcal{A}$ of F_N we define in a similar spirit, for any T which belongs to $\partial \mathrm{cv}_N$, the laminary languages

$$\mathcal{L}^\varepsilon_\mathcal{A}(T) = \bigcup_{w\in\Omega_\varepsilon(T)} \mathcal{L}_\mathcal{A}(L^2(w))$$

and

$$\mathcal{L}^\Omega_\mathcal{A}(T) = \bigcap_{\varepsilon>0} \mathcal{L}^\varepsilon_\mathcal{A}(T)\,.$$

Hence $u \in F(\mathcal{A})$ belongs to $\mathcal{L}^\Omega_\mathcal{A}(T)$ if and only if for all $\varepsilon > 0$ there exists a cyclically reduced word $w \in F(\mathcal{A})$ with $\|w\|_T < \varepsilon$ and an exponent $m \geq 1$ such that u is a subword of w^m. However, it suffices to consider exponents which satisfy $m \leq |u|_\mathcal{A}$ (= the word length of u in $\mathcal{A}^{\pm 1}$). Thus we obtain as direct consequence:

Remark 4.2. A word $u \in F(\mathcal{A})$ belongs to $\mathcal{L}^\Omega_\mathcal{A}(T)$ if and only if for all $\varepsilon > 0$ there exists a cyclically reduced word $w \in F(\mathcal{A})$ with $\|w\|_T < \varepsilon$ such that u is a subword of w.

It follows easily that these laminary languages correspond precisely to the algebraic laminations $L^2_\varepsilon(T)$ and $L^2_\Omega(T)$, under the bijection $\rho^\mathcal{A}_\mathcal{L}\rho^2_\mathcal{A}$ established in Theorem 1.1 of [CHL-I]. Indeed, except for the passage from $\Omega^2_\varepsilon(T)$ to $L^2_\varepsilon(T)$, i.e. closing up in $\partial^2 F_N$, the identity between the corresponding laminary languages is definitory. But as the language $\mathcal{L}^\varepsilon_\mathcal{A}(T)$ is already laminary (see §5 of [CHL-I]), it follows that it agrees with the laminary language $\rho^\mathcal{A}_\mathcal{L}\rho^2_\mathcal{A}(L^2_\varepsilon(T))$ associated canonically to the closure of $\Omega^2_\varepsilon(T)$ in $\partial^2 F_N$.

We finish this section with two observations regarding the dual lamination for particular cases or $\mathbb{R}$-trees T. Both of the following remarks follow directly from our definitions above.

Remark 4.3. Let $T \in \overline{\text{cv}}_N$ be an $\mathbb{R}$-tree. For any point $x \in T$ we consider the stabilizer

$$\text{Stab}(x) = \{w \in F_N \mid wx = x\} .$$

Then every $w \in \text{Stab}(x)$ is conjugate to a word in $\mathcal{L}^{\Omega}_{\mathcal{A}}(T)$, for any basis $\mathcal{A}$ of F_N.

Equivalently, noting that $\text{Stab}(x)$ is a free group F_k of rank $k \leq N$ (see [GL95]), the lamination $L^2_\Omega(T)$ contains the image (under the map canonically induced by the inclusion $F_k \subset F_N$, see [CHL-I], Remark 8.1) of the full lamination $L^2_\Omega(F_k) = \partial^2 F_k$.

The following remark is useful with respect to the canonical decomposition of an $\mathbb{R}$-tree $T \in \overline{\text{cv}}_N$ into T_d and T_c as given in Remark 2.1: If either of them is non-empty, one can contract the connected components of the other one to get F_N-equivariant, distance decreasing maps $T \to T/T_d$ or $T \to T/T_c$, and both quotient $\mathbb{R}$-trees belong again to $\overline{\text{cv}}_N$.

Remark 4.4. Let $T, T' \in \overline{\text{cv}}_N$, and let $T \to T'$ be an F_N-equivariant, distance decreasing map. Then one has:

$$L^2_\Omega(T) \subset L^2_\Omega(T') .$$

These two observations are the starting point for a more detailed structural analysis of F_N-actions on $\mathbb{R}$-trees: More details will be given in [CHL3].

5 One-sided infinite words

Let T be, as before, an $\mathbb{R}$-tree with a left action by isometries of the free group F_N. We fix a basis $\mathcal{A}$ of F_N and a point P of T. The choice of the basis $\mathcal{A}$ gives us an identification between the boundary ∂F_N, and the space $\partial F(\mathcal{A})$ of (one-sided) infinite reduced words $x_1 x_2 x_3 \ldots$ in $\mathcal{A}^{\pm 1}$.

Following [LL03] we denote by $L^1_{\mathcal{A}}(T) \subset \partial F(\mathcal{A})$ the subset of those infinite reduced words $X = x_1 x_2 \ldots$ in $\mathcal{A}^{\pm 1}$ which have the property that for some $P \in T$ the sequence $(X_i P)_{i \in \mathbb{N}}$ is bounded, where X_i is the prefix of length i of X. We observe:

Remark 5.1. (1) If X belongs to $L^1_{\mathcal{A}}(T)$, then for *any* $P \in T$ the sequence $(X_i P)_{i \in \mathbb{N}}$ is bounded.

(2) If X does not belong to $L^1_{\mathcal{A}}(T)$, then for any $C > 0$, any $P \in T$ and any integer K there exist $l > k \geq K$ such that $d(X_k P, X_l P) > C$.

(3) If in addition T satisfies BBT, then for any $X = x_1 x_2 \ldots \in L^1_{\mathcal{A}}(T)$ there exists an integer $K \geq 0$ such that for all $k, l \geq K$ one has $d(X_k P, X_l P) = d(x_{k+1} \ldots x_l P, P) < 3BBT(\mathcal{A}, P)$.

Proposition 5.2 ([LL03]). *The canonical identification $\partial F(\mathcal{A}) = \partial F_N$ associates to the subset $L^1_{\mathcal{A}}(T) \subset \partial F(\mathcal{A})$ a set $L^1(T) \subset \partial F_N$ that does not depend on the choice of $\mathcal{A}$ (which justifies the notation $L^1(T)$).*

Proof. Let $\mathcal{B}$ be another basis for F_N and X be in ∂F_N. Denote by $X_{\mathcal{A}}$ and $X_{\mathcal{B}}$ the corresponding (one-sided) infinite reduced words in $\partial F(\mathcal{A})$ and $\partial F(\mathcal{B})$. The prefix sequences $(X_{\mathcal{A},i})_{i\in\mathbb{N}}$ and $(X_{\mathcal{B},i})_{i\in\mathbb{N}}$ (which are sequences of elements of F_N) both converge to X. Geometrically they are two quasi-geodesics in F_N. If we fix the word metric $d_{\mathcal{A}}$ on F_N, then the first sequence lies on a geodesic, and the second on a quasi-geodesic. In particular, as follows from Cooper's cancellation bound (see Lemma 7.1 of [CHL-I]), the two sequences have the property that for any positive integer j there exists a positive integer i such that $d_{\mathcal{A}}(X_{\mathcal{B},j}, X_{\mathcal{A},i}) < C$, where $C = \mathrm{BBT}(\mathcal{B}, \mathcal{A})$ is Cooper's cancellation bound between basis $\mathcal{A}$ and $\mathcal{B}$.

This shows that the sequence of points $X_{\mathcal{A},i}P$ is of bounded diameter if and only if the sequence of the $X_{\mathcal{B},j}P$ is. □

Let us now state some properties of $L^1(T)$: Note first that the last proposition, and thus the definition of $L^1(T)$, does not require that T satisfies BBT. Furthermore, the subset $L^1(T)$ of ∂F_N is F_N-invariant, and, unless it is empty, it is dense in ∂F_N (as is any F_N-invariant non-empty subset of ∂F_N). The set $L^1(T)$ is empty if and only if the action of F_N on T is free and discrete.

Using the projection on the first coordinate, $\pi : \partial^2 F_N \to \partial F_N, (X, X') \mapsto X$, one can alternatively deduce from the algebraic lamination $L^2_\Omega(T)$ defined in section 4 a second F_N-invariant subset of ∂F_N associated to T, namely the image set $\pi(L^2_\Omega(T))$.

Proposition 5.3. *For any $\mathbb{R}$-tree T with isometric F_N-action that satisfies BBT, one has:*

$$\pi(L^2_\Omega(T)) \subset L^1(T)$$

Proof. Let $\mathcal{A}$ be a basis of F_N and $X = x_1x_2\ldots$ a reduced infinite word in $\partial F(\mathcal{A}) = \partial F_N$. If X does not belong to $L^1(T)$, then it follows from Remark 5.1 (2) that for any point $P \in T$, for any $C > 0$ and for any $K > 0$, there exist k, l with $K \leq k \leq l$ such that $d_{\mathcal{A}}(X_kP, X_lP) > C$ where X_i is the prefix of X of length i. It follows directly from Lemma 3.1 (c) that, chosing C large enough (depending on the given P and $\mathcal{A}$), the word $X_k^{-1}X_l$ cannot be a subword of any cyclically reduced word that represents an element $w \in \Omega_\varepsilon(T)$, for small $\varepsilon > 0$. Hence $X_k^{-1}X_l$ and its inverse $X_l^{-1}X_k$ do not belong to $\mathcal{L}^\Omega_{\mathcal{A}}(T)$, by Remark 4.2.

But if there is some $X' = x'_1x'_2\ldots \in \partial F_N$ such that (X, X') belongs to $L^2_\Omega(T)$, then the reduced biinfinite word $\rho^2_{\mathcal{A}}(X, X') = X^{-1}X' = \ldots x^{-1}_{j+2}x^{-1}_{j+1} \cdot x'_{j+1}x'_{j+2}\ldots$ in the symbolic lamination $\rho^2_{\mathcal{A}}(L^2_\Omega(T)) = L^\Omega_{\mathcal{A}}(T) \subset \Sigma_{\mathcal{A}}$ associated to $L^2_\Omega(T)$ (compare Proposition 4.4 of [CHL-I]) will contain any of these $X_l^{-1}X_k = x_l^{-1}\ldots x^{-1}_{k+1}$ with sufficiently large $k < l$ as subword, in contradiction to the statement at the end of the last paragraph. This shows that X cannot belong to $\pi(L^2_\Omega(T))$. □

The converse inclusion with respect to Proposition 5.3 does not hold in general, as will be seen in section 7. In fact, one has to regard $L^1(T)$ as a finer invariant of T than the algebraic lamination $L^2_\Omega(T)$, which only depends on a weakened version of the topology of T, compare [CHL05], while $L^1(T)$ may change when different $\mathbb{R}$-tree structures are varying on a given topological tree T. For more details see [CHLL]. The fact that one can derive $L^2_\Omega(T)$ from $L^1(T)$ will be shown below: it is a direct consequence of Proposition 5.6.

For any basis $\mathcal{A}$ of F_N and any set $S \subset \partial F_N$ we denote by $\mathcal{L}^\infty_\mathcal{A}(S)$ the the set of words $u \in F(\mathcal{A})$ and their inverses such that u occurs infinitely often as subword in some of the reduced infinite words $X_\mathcal{A}$ that represent an element $X \in S$ (we say u is *recurrent* in $X_\mathcal{A}$). For any non-empty S the language $\mathcal{L}^\infty_\mathcal{A}(S)$ is laminary (because we artificially added to $\mathcal{L}^\infty_\mathcal{A}(S)$ the inverses of any recurrent u): we call it the *recurrent laminary language in* $\mathcal{A}^{\pm 1}$ *associated to* S.

Definition 5.4. For any basis $\mathcal{A}$ of F_N and any non-empty set $S \subset \partial F_N$ let $L^2_\infty(S)$ the algebraic lamination defined by the recurrent laminary language in $\mathcal{A}^{\pm 1}$ associated to S:

$$L^2_\infty(S) = L^2(\mathcal{L}^\infty_\mathcal{A}(S))$$

Here (and below) we denote by $L^2(\mathcal{L})$ the algebraic lamination $(\rho^\mathcal{A}_\mathcal{L}\rho^2_\mathcal{A})^{-1}(\mathcal{L})$ defined in [CHL-I] for any laminary language $\mathcal{L}$.

Proposition 5.6 below justifies the absence of mentioning explicitely the basis $\mathcal{A}$ in the notation $L^2_\infty(S)$. But first we observe:

For any basis $\mathcal{A}$ of F_N, any $k \in \mathbb{N}$ and any $w \in F(\mathcal{A})$, we denote by $w\dagger_k \in F(\mathcal{A})$ the subword of w obtained from chopping off the initial and final subword of length k, see [CHL-I], §5. For any second basis $\mathcal{B}$ of F_N consider Cooper's cancellation bounds $C = \mathrm{BBT}(\mathcal{B}, \mathcal{A})$ and $C' = \mathrm{BBT}(\mathcal{A}, \mathcal{B})$ (as given in [CHL-I], §7), as well as $C'' = C' + \lambda C$, where λ is the maximal length of the elements of $\mathcal{A}$ written as words in $\mathcal{B}^{\pm 1}$.

Lemma 5.5. *Consider a word $u_\mathcal{B} \in F(\mathcal{B})$ in $\mathcal{B}^{\pm 1}$, and let $u_\mathcal{A} \in F(\mathcal{A})$ be the word in $\mathcal{A}^{\pm 1}$ representing the same element of F_N. Let $v_\mathcal{A} = u_\mathcal{A}\dagger_C$, and let $w_\mathcal{A} \in F(\mathcal{A})$ be any word that contains $v_\mathcal{A}$ as a subword. Let $w_\mathcal{B} \in F(\mathcal{B})$ be the word in $\mathcal{B}^{\pm 1}$ representing the same element of F_N as $w_\mathcal{A}$. Then $u_\mathcal{B}\dagger_{C''}$ is a subword of $w_\mathcal{B}\dagger_{C'}$.*

Proof. This follows from a straight forward calculation. ☐

Proposition 5.6. *Let S be a non-empty subset of ∂F_N, and $\mathcal{A}$ and $\mathcal{B}$ be two basis of F_N. Then one has:*

$$L^2(\mathcal{L}^\infty_\mathcal{A}(S)) = L^2(\mathcal{L}^\infty_\mathcal{B}(S))\,.$$

Proof. It suffices to prove for any $Y \in S$ the equality

$$L^2(\mathcal{L}^\infty_\mathcal{A}(Y)) = L^2(\mathcal{L}^\infty_\mathcal{B}(Y)).$$

Thus, for any (X, X') in $L^2(\mathcal{L}^\infty_\mathcal{A}(Y))$, we have to prove that the biinfinite word $Z_\mathcal{B} = \rho_\mathcal{B}(X, X')$ in $\mathcal{B}^{\pm 1}$ is contained in the symbolic lamination $L(\mathcal{L}^\infty_\mathcal{B}(Y)) = (\rho^\mathcal{B}_\mathcal{L})^{-1}(\mathcal{L}^\infty_\mathcal{B}(Y))$ (see [CHL-I], §§4 and 5). Let $u_\mathcal{B} \in F(\mathcal{B})$ be a finite subword of $Z_{\mathcal{B}}$, and let $u_{\mathcal{A}} \in F(\mathcal{A})$ be the word in $\mathcal{A}^{\pm 1}$ representing the same element of F_N as $u_{\mathcal{B}}$. Using Cooper's cancellation bound $C = \mathrm{BBT}(\mathcal{B}, \mathcal{A})$, we see that $v_{\mathcal{A}} = u_{\mathcal{A}} \dagger_C$ is a subword of the biinfinite word $Z_{\mathcal{A}} = \rho_{\mathcal{A}}(X, X')$. By definition of $L^2(\mathcal{L}^{\infty}_{\mathcal{A}}(Y))$, $v_{\mathcal{A}}$ is recurrent in the infinite word $Y_{\mathcal{A}} \in \partial F(\mathcal{A})$ representing Y. Let $w_{\mathcal{A}}$ be a subword of $Y_{\mathcal{A}}$ such that $v_{\mathcal{A}}$ is a subword of $w_{\mathcal{A}}$, sufficiently far from the beginning and the end of $w_{\mathcal{A}}$.

Let $w_{\mathcal{B}} \in F(\mathcal{B})$ be the word in $\mathcal{B}^{\pm 1}$ representing the same element of F_N as $w_{\mathcal{A}}$. Using Cooper's cancellation bound $C' = \mathrm{BBT}(\mathcal{A}, \mathcal{B})$, we get that $w_{\mathcal{B}} \dagger_{C'}$ is a subword of the infinite word $Y_{\mathcal{B}} \in \partial F(\mathcal{B})$ representing Y, and that $u_{\mathcal{B}} \dagger_{C''}$ is a subword of $w_{\mathcal{B}} \dagger_{C'}$, for $C'' > 0$ depending only on $\mathcal{A}$ and $\mathcal{B}$ as specified in Lemma 5.5. This proves that $u_{\mathcal{B}} \dagger_{C''}$ is recurrent in $Y_{\mathcal{B}}$. As $u_{\mathcal{B}} \in \mathcal{L}(Z_{\mathcal{B}})$ was chosen arbitrarily and $\mathcal{L}(Z_{\mathcal{B}})$ is a laminary language, this proves that the biinfinite word $Z_{\mathcal{B}} = \rho_{\mathcal{B}}(X, X')$, for any $(X, X') \in L^2(\mathcal{L}^{\infty}_{\mathcal{A}}(Y))$, is contained in the symbolic lamination $L(\mathcal{L}^{\infty}_{\mathcal{B}}(Y))$. Thus we have

$$L^2(\mathcal{L}^{\infty}_{\mathcal{A}}(Y)) \subset L^2(\mathcal{L}^{\infty}_{\mathcal{B}}(Y)),$$

which proves the proposition. □

Definition 5.7. Let T be an $\mathbb{R}$-tree with very small action of F_N with dense orbits, and let $\mathcal{A}$ be any basis of F_N. We define $L^2_{\infty}(T)$ as the algebraic lamination defined by the recurrent language associated to $L^1(T)$:

$$L^2_{\infty}(T) = L^2(\mathcal{L}^{\infty}_{\mathcal{A}}(L^1(T)))$$

It follows from Proposition 5.2 and Proposition 5.6 that $L^2_{\infty}(T)$ does not depend on the choice of the basis $\mathcal{A}$.

Proposition 5.8. *Let T be an $\mathbb{R}$-tree with very small action of F_N with dense orbits. The lamination $L^2_{\Omega}(T)$ of Definition 4.1 and the lamination $L^2_{\infty}(T)$ of Definition 5.7 are equal:*

$$L^2_{\Omega}(T) = L^2_{\infty}(T).$$

Proof. Let $\mathcal{A}$ be a basis of F_N. We will prove that the laminary languages associated to these laminations (via the canonical map $\rho^{\mathcal{A}}_{\mathcal{L}} \rho^2_{\mathcal{A}} : \Lambda^2(F_N) \to \Lambda_{\mathcal{L}}(\mathcal{A})$ from [CHL-I]) are equal.

We first prove that $\mathcal{L}^{\infty}_{\mathcal{A}}(L^1(T)) \subset \mathcal{L}^{\Omega}_{\mathcal{A}}(T)$: Since the F_N-action on T is very small and has dense orbits, for any $\varepsilon > 0$ (fixed for the rest of this proof) there exists by Lemma 3.2 a basis $\mathcal{B}$ of F_N and a point $P \in T$ such that $\mathrm{BBT}(\mathcal{B}, P) < \varepsilon$ and $\mathrm{vol}(\mathcal{B}, P) < \varepsilon$.

Let $X = x_1 x_2 \ldots$ be a reduced infinite word in $\mathcal{A}^{\pm 1}$ that belongs to $L^1_{\mathcal{A}}(T)$. We know that, according to the above definition of the associated recurrent laminary language, if $X_{k,l} = x_k \ldots x_l \in \mathcal{L}^{\infty}_{\mathcal{A}}(X)$, then there are arbitrary large k', l' with $X_{k,l} = X_{k',l'}$. Hence Cooper's cancellation bound, see §7 of [CHL-I], assures us that, when writing X as a reduced word in the basis $\mathcal{B}$, say $X_{\mathcal{B}} = y_1 y_2 \ldots$, there is a recurrent subword $Y_{r,s} = y_r \ldots y_s$ of $X_{\mathcal{B}}$ which has the property that, when written as word $Y_{\mathcal{A}}$ in $\mathcal{A}^{\pm 1}$, the latter contains $X_{k,l}$ as subword. On the other hand, since $\mathcal{B}$ contains at least two elements (by the general assumption that the rank N of F_N is at least 2), there is a $y \in \mathcal{B}^{\pm 1}$ such that $Y_{r,s}y$ is cyclically reduced. Hence we obtain from Remark 5.1 (3), from Lemma 3.1 (d) and from the above assumptions $\mathrm{BBT}(\mathcal{B}, P) < \varepsilon$ and $\mathrm{vol}(\mathcal{B}, P) < \varepsilon$ that

$$\|Y_{r,s}y\|_T \leq d(P, Y_{r,s}P) + d(P, yP) \leq 4\mathrm{BBT}(\mathcal{B}, P) < 4\varepsilon\,.$$

But for s sufficiently large and r sufficiently small the subword $X_{k,l}$ of $Y_{\mathcal{A}}$ will not be cancelled, when $Y_{r,s}y$ is written in $\mathcal{A}^{\pm 1}$ and subsequently cyclically reduced, by Lemma 5.5. This implies that $X_{k,l}$ belongs to $\mathcal{L}_{\mathcal{A}}^{4\varepsilon}(T)$, which proves the assertion.

We now turn to the proof of the converse inclusion, namely $\mathcal{L}_{\mathcal{A}}^{\Omega}(T) \subset \mathcal{L}_{\mathcal{A}}^{\infty}(L^1(T))$: For any word w in $F(\mathcal{A})$ we distinguish between its *conjugating part* $v \in F(\mathcal{A})$ and its *cyclically reduced part* $w' \in F(\mathcal{A})$, where $w = vw'v^{-1}$ is in reduced form, with w' cyclically reduced.

Let $u \in \mathcal{L}_{\mathcal{A}}^{\Omega}(T)$ be a word in $\mathcal{A}^{\pm 1}$, P a point in T and $\varepsilon > 0$. We first want to show that there exists a word $w \in F(\mathcal{A})$ that contains u as a subword of its cyclically reduced part and satisfies $d(P, wP) < \varepsilon$.

Indeed, by Remark 4.2 there exists a cyclically reduced word w in $F(\mathcal{A})$ of which u is a subword and such that $\|w\|_T < \frac{\varepsilon}{6}$. Then u is a subword of any cyclic conjugate of w^2. As the action of F_N on T has dense orbits there exists a word v of $F(\mathcal{A})$ such that

$$d(v^{-1}P, \mathrm{Ax}(w)) < \frac{\varepsilon}{3}\,.$$

An easy calculation then shows that vw^2v^{-1} satisfies $d(P, vw^2v^{-1}P) < \varepsilon$, which is what we claimed.

Thus there exists a sequence of words $u_k \in F(\mathcal{A})$ where each of them contains u or u^{-1} as a subword of its cyclically reduced part, such that $d(P, u_kP) < \frac{1}{2^k}$. We apply Lemma 5.9 stated below to obtain a sequence of $w_n = u_{k_n}^{d_n}$, with $d_n = \pm 1$, and with the further property that in each of the products w_nw_{n+1} the cancellation in w_nw_{n+1} does not go further than the conjugating parts of w_n and w_{n+1}. Then $X = w_1w_2w_3\cdots$ is a word in $L_{\mathcal{A}}^1(T)$ in which u or u^{-1} is recurrent.

This concludes the proof of Proposition 5.8, modulo the proof of the subsequent lemma. □

Lemma 5.9. *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of words from $F(\mathcal{A})$. Then there is an infinite subsequence $(u_{k_n})_{n\in\mathbb{N}}$ and exponents $d_n = \pm 1$, such that the sequence of words $w_n = u_{k_n}^{d_n}$ possesses the additional property that in each of the products w_nw_{n+1} the cancellation in w_nw_{n+1} does not exceed the conjugating parts of either w_n or w_{n+1}.*

Proof. If the given words u_k are almost all cyclically reduced, then we can build the sequence w_n by chosing inductively $w_{n+1} = u_{n+1}$ or $w_{n+1} = {u_{n+1}}^{-1}$ according to the previous choice to avoid any cancellation in w_nw_{n+1}.

If the given sequence $(u_k)_{k\in\mathbb{N}}$ contains a subsequence of words u_{k_n} with bounded length of their conjugating part, then there exists a subsequence with constant conjugating part and we can use the previous construction.

In the remaining case there exists a subsequence of words u_{k_n} with strictly increasing length of their conjugating part. Assume that $w_1, \ldots, w_n$ were already chosen, $w_i = u_{k_i}^{d_i}$ with $d_i = \pm 1$, and with the property that the cancellation in $w_i w_{i+1}$ is not more than the conjugating part of w_i and strictly less than the conjugating part of w_{i+1}. Then replacing the last word w_n by its inverse w_n^{-1} does not change this property. If the cancellation in $w_n u_{k_{n+1}}$ is bigger than the conjugating part of w_n we replace w_n by its inverse w_n^{-1}. It follows that the cancellation in both $w_n u_{k_{n+1}}$ and $w_n u_{k_{n+1}}^{-1}$ is then not more than the conjugating part of w_n and strictly less than the conjugating part of $u_{k_{n+1}}$, as the length of the latter is strictly bigger than that of the conjugating part of w_n (by our original "strictly increasing" condition for this case). □

Remark 5.10. The precise relationship between the various F_N-invariant sets of one-sided infinite words associated to a lamination or to an $\mathbb{R}$-tree is rather intricated, and it seems difficult to express the algebraic lamination associated to an $\mathbb{R}$-tree properly in terms of such a set. An attempt, however confusing or misleading it may be, is made in the subsequent paragraph:

For any algebraic lamination L^2 we denote by $L^1_\infty(L^2) \subset \partial F_N$ the set of all infinite words $X = x_1 x_2 \ldots$ in some basis $\mathcal{A}$ of F_N, which have the property that their associated recurrent laminary language is contained in $\mathcal{L}_\mathcal{A}(L^2)$. One can use Cooper's cancellation bound (or rather a variation of Proposition 5.6) to show that $L^1_\infty(L^2)$ does not depend on the choice of $\mathcal{A}$. We believe that Proposition 5.3 can be extended to show that $\pi(L^2)$ is always a subset of $L^1_\infty(L^2)$. The converse, however, seems in general to be wrong: For example, for $\mathbb{R}$-tree laminations $L^2_\Omega(T)$, where T satisfies BBT and has dense orbits, we know that both of the inclusions

$$\pi(L^2_\Omega(T)) \subset L^1(T) \subset L^1_\infty(L^2_\Omega(T))$$

hold, but we strongly suspect that, for any such T, they both are proper inclusions. On the other hand, we have seen above that the three recurrent laminary languages associated to these three F_N-invariant sets are equal.

The set $L^1_\infty(L^2_\Omega(T))$ deserves some further attention, since it depends not on the metric on T, but only on the *observer's topology* on T, compare [CHL05]. Contrary to what seems to be indicated by the results of [CHL05], the set $L^1_\infty(L^2_\Omega(T))$ shows that the lamination $L^2(T)$ alone suffices, without invoking the metric on T, to exhibit certain completion points of the topological tree T as lying "far out at infinity".

6 Bounded Back Tracking property and the map $\mathcal{Q}$

Throughout this section we assume that T satisfies the property BBT (see §2), which follows for example if T is small. It is an easy exercise (compare [GJLL98]) to show that the property BBT ensures that every element $X \in \partial F_N \smallsetminus L^1(T)$ determines, through picking any point $P \in T$ and any sequence of elements $X_i \in F_n$ that converges towards X, a well defined point

$$\mathcal{Q}(X) = \lim_{i \to \infty} X_i P$$

of the Gromov boundary ∂T of T.

We suppose from now on that T is an $\mathbb{R}$-tree with very small F_N-action with dense orbits, and that for some (arbitrary) point $P \in T$ one has given a sequence of bases $\mathcal{A}_k$ of F_N that satisfies the properties assured by Lemma 3.2: Both $\mathrm{vol}(\mathcal{R}_{\mathcal{A}_k})$ and $\mathrm{BBT}(\mathcal{A}_k)$ tend to 0, for $k \to \infty$.

For any infinite reduced word $X = x_1 x_2 \ldots \in \partial F(\mathcal{A}_k)$ that represents an element of $L^1(T)$, the sequence of points $(x_1 \ldots x_i P)_{i \in \mathbb{N}}$ eventually stays in a bounded region $R(X, k)$ of diameter $3\mathrm{BBT}(\mathcal{A}_k)$ (compare Remark 5.1 (c)), so that we can associate to X a well defined point $\mathcal{Q}(X) = \lim_{k \to \infty} R(X, k)$. It has been shown in [LL03] that $\mathcal{Q}(X)$ depends only on $X \in \partial F_N$ and not on the above choice of P and of the $\mathcal{A}_k$. It is important to note that $\mathcal{Q}(X)$ may well be contained in the metric completion $\overline{T}$ of T, but not in T itself.

Alternative definitions of the point $\mathcal{Q}(X)$, for any $X \in \partial F_N$, which do not need to consider an infinite change of bases of F_N, have been given in [LL] and in[LL03], Lemma 3.4:

Lemma 6.1. *For all X in ∂F_N, for any sequence of points $X_i \in F_N$ which converge towards X, and for any point P of T, one has in $\overline{T} \cup \partial T$:*

$$[P, \mathcal{Q}(X)] = \overline{\bigcup_{n \in \mathbb{N}} \bigcap_{i \geq n} [P, X_i P]}\,.$$

Lemma 6.2. *For all X in ∂F_N and for all P in T, the point $\mathcal{Q}(X)$ is the only point of $\overline{T} \cup \partial T$ such that there exists a sequence of elements $X_i \in F_n$ which converge towards X and a point P in T such that the points $X_i P$ converge to $\mathcal{Q}(X)$.*

From Lemma 6.1 and Remark 5.1 (3) it follows directly:

Lemma 6.3. *Let P be a point in an $\mathbb{R}$-tree $T \in \partial cv_N$ with dense F_N-orbits, and let $\mathcal{A}$ be a basis of F_N. Then for every $X = x_1 x_2 \ldots \in L^1_{\mathcal{A}}(T)$ there exists a bound $K \geq 0$ such that for every $X_k = x_1 \ldots x_k$ with $k \geq K$ one has:*

$$d(X_k P, \mathcal{Q}(X)) \;\leq\; 3\, BBT(\mathcal{A}, P)\,.$$

Summarizing the above discussion, we observe that for every very small $\mathbb{R}$-tree T with dense orbits every boundary point $X \in \partial F_N \smallsetminus L^1(T)$ determines a point $\mathcal{Q}(X) \in \partial T$, while $X \in L^1(T)$ determines a point $\mathcal{Q}(X)$ in T or in its metric completion $\overline{T}$. This defines a map

$$\mathcal{Q} : \partial F_N \to \overline{T} \cup \partial T,\ X \mapsto \mathcal{Q}(X)$$

which is F_N-equivariant and surjective [LL03], but a priori it is injective and continuous (with respect to the canonical boundary topologies) only on $\partial F_N \smallsetminus L^1(T)$. In particular, on $L^1(T)$ the map $\mathcal{Q}$ is not continuous with respect to the metric on T (it does though possess on $L^1(T)$ a kind of lower semi-continuous property, see Proposition 3.8 of [LL03]). In [CHL05] it is proved that $\mathcal{Q}$ is continuous if we replace the metric topology on T by the weaker *observer's topology* (this is the topology for which a basis of open subsets is given by connected components of $T \smallsetminus \{P\}$ for all points P of T).

The basic phenomenon for the lack of continuity of the map $\mathcal{Q}$ is illustrated as follows: If X_k is a converging sequence of elements from ∂F_N with the property that for some point $Q \in T$ the segments $[Q, \mathcal{Q}(X_k)]$ have pairwise intersections of length converging to 0, then $X = \lim X_k$ satisfies $Q = \mathcal{Q}(X)$, while the lengths of the segments $[Q, \mathcal{Q}(X_k)]$ may well not converge to 0 or even converge to ∞.

However, one can prove that the map $\mathcal{Q}$ has the "closed graph property" (which will not be used in the sequel):

Remark 6.4. Let T be an $\mathbb{R}$-tree in $\partial\mathrm{cv}_N$ which has dense F_N-orbits, and, consider a sequence of boundary points $X_k \in \partial F_N$ that converge to some $X \in \partial F_N$. Assume that the image points $\mathcal{Q}(X_k) \in \overline{T} \cup \partial T$ converge to a point $R \in \overline{T} \cup \partial T$. Then one has:

$$R = \mathcal{Q}(X).$$

7 Geodesic lamination on a surface

To gain some geometric intuition, let us consider in this section the special case of an $\mathbb{R}$-tree that is dual to a measured lamination in a surface: As in §3 of [CHL-I] we denote by S a surface with non-empty boundary and with negative Euler characteristic, provided with a hyperbolic structure. The latter is given by an identification of the universal covering $\widetilde{S}$ with a convex part of the hyperbolic plane $\mathbb{H}^2$, which realizes the deck transformation action of $F_N = \pi_1 S$ on $\widetilde{S}$ by hyperbolic isometries. Then any geodesic lamination $\mathfrak{L}$ on S defines, by taking the full preimage, a geodesic lamination $\widetilde{\mathfrak{L}}$ in $\widetilde{S} \subset \mathbb{H}^2$, on which $F_N = \pi_1 S$ acts canonically.

There is a canonical *dual tree* $T_{\mathfrak{L}}$ with F_N-action by homeomorphisms associated to $\mathfrak{L}$ (or to $\widetilde{\mathfrak{L}}$), which is defined by associating to every non-boundary leaf of $\widetilde{\mathfrak{L}}$ a point of $T_{\mathfrak{L}}$ which is not a branch point, and to the closure of any complementary component of $\widetilde{\mathfrak{L}}$ in $\widetilde{S}$ a branch point of $T_{\mathfrak{L}}$. This association is a bijective and can be made continuous: If the lamination $\mathfrak{L}$ is finite, then $T_{\mathfrak{L}}$

is simplicial, so that there is no ambiguity. If $\mathfrak{L}$ is infinite, then defining properly the topology of $T_{\mathfrak{L}}$ is much more delicate; we refer the interested reader to [CHL05] where this problem has been dealt with properly.

We now assume that the lamination $\mathfrak{L}$ is provided with a transverse measure μ (see [FLP91]). Then the lift $\widetilde{\mu}$ of μ to $\widetilde{\mathfrak{L}}$ gives rise to a metric on $T_{\mathfrak{L}}$ by defining for any points $x, y \in T_{\mathcal{L}}$, corresponding to leaves $l_x, l_y \in \widetilde{\mathfrak{L}}$, the distance $d(x, y) = \widetilde{\mu}(\alpha)$, where α is an arc in $\widetilde{S}$ with one endpoint on l_x and the other on l_y, and α is assumed to be geodesic in $\mathbb{H}^2$ and hence transverse to $\widetilde{\mathfrak{L}}$. This makes $T_{\mathfrak{L}}$ into an $\mathbb{R}$-tree T_μ with isometric F_N-action. It is noteworthy that, in the exceptional but fascinating case where $\mathfrak{L}$ is not uniquely ergodic, projectively different transverse measures μ will produce projectively distinct $\mathbb{R}$-trees T_μ, and that the simplex of projective measures on $\mathcal{L}$ (located on the Thurston boundary of Teichmüller space) gives rise to an anologous simplex of $\mathbb{R}$-trees in ∂CV_n.

We now consider an arbitrary point Q on the boundary $\partial\widetilde{S} \subset S^1_\infty = \partial\mathbb{H}^2$, where $\partial\widetilde{S}$ also coincides via our above identification $F_N = \pi_1 S$ with the Gromov boundary ∂F_N. Let β be the geodesic arc which connects some arbitrary chosen point P in $\widetilde{S}$ to Q. We distinguish three cases:

1. Q is the endpoint of a leaf l of $\widetilde{\mathfrak{L}}$. Then $\widetilde{\mu}(\beta)$ is finite. In fact, β projects to a segment in T_μ of length $\widetilde{\mu}(\beta)$. Denote by $\widehat{Q} \in T_\mu$ the image of Q under this projection.

2. The measure $\widetilde{\mu}(\beta)$ is infinite. Then β projects to an infinite arc in T_μ, and Q defines a point $\widehat{Q}$ in the Gromov boundary ∂T_μ of T_μ (which is independent of the choice of β).

3. In the remaining case the point Q defines a point $\widehat{Q}$ in the metric completion $\overline{T}_\mu$ of T_μ, and the arc β projects to a finite open arc in T_μ which becomes closed when adding the point $\widehat{Q}$.

In each of the three cases the geometrically described point $\widehat{Q}$ is precisely the image $\mathcal{Q}(Q)$ of the point Q, if Q is viewed as element of ∂F_N via the above identification $\partial\widetilde{S} = \partial F_N$.

We would like to note that this third class is non-empty for many laminations $\mathfrak{L}$: For example, if $\mathfrak{L}$ is the contracting (or expanding) lamination fixed by a pseudo-Anosov automorphism φ of S, then it suffices to consider a lift $\widetilde{\varphi}$ of φ to $\widetilde{S}$ that does not fix any leaf (or permute finitely many leaves) of $\widetilde{\mathfrak{L}}$. The repulsive fixed point of $\widetilde{\varphi}$ on $\partial\widetilde{S}$ will then define such a point Q. Note that the existence of such lifts $\widetilde{\varphi}$ of φ has been proved in [LL00].

The distinction of these three cases is illuminating in that it shows that the set $L^1(T_\mu)$, given here by the cases 1 and 3, may well be strictly larger than the F_N-invariant set $\pi(L^2(T_\mu)) \subset \partial F_N$ canonically associated to $L^2(T_\mu)$, given here by case 1. Indeed, while $L^2(T_\mu)$ (and accordingly, the occurence of the case 1 above) depends only on $\mathfrak{L}$ and not on μ, we do not know whether (but suspect that) the partition of the complement into cases 2 and 3 may actually depend on μ.

The distinction of $L^1(T_\mu)$ into cases 1 and 3 was the original motivation behind the definition of the algebraic lamination $L^2(T)$ given in this paper.

8 The lamination $L_{\mathcal{Q}}(T)$ and the map $\mathcal{Q}^2$

By restricting the domain and the range of the map $\mathcal{Q}$ introduced in section 6, one obtains the map:

$$\mathcal{Q}^1 : L^1(T) \to \overline{T},\ X \mapsto \mathcal{Q}(X)\,.$$

Recall from §6 that the map $\mathcal{Q}^1$ is surjective (see [LL03]), but in general not injective. Unless $\mathcal{Q}(X)$ has a non-trivial stabilizer in F_N, the map $\mathcal{Q}^1$ is conjectured to be finite to one (see Remark 3.6 of [LL03]).

Definition 8.1. To every very small $\mathbb{R}$-tree T with dense orbits we associate the following F_N- and flip-invariant subset of $\partial^2 F_N$:

$$L^2_{\mathcal{Q}}(T) = \{(X, X') \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(X')\}.$$

Note that, as $\mathcal{Q}$ is injective on $\partial F_N \smallsetminus L^1(T)$, for (X, X') in $L^2_{\mathcal{Q}}(T)$ one has that X and X' belong to $L^1(T)$. Equivalently, we know that $\mathcal{Q}(X) = \mathcal{Q}(X')$ lies in $\overline{T}$ and not in ∂T.

Definition 8.2. We define a map

$$\mathcal{Q}^2 : L^2_{\mathcal{Q}}(T) \to \overline{T},\ (X, X') \mapsto \mathcal{Q}(X) = \mathcal{Q}(X')$$

which is F_N-equivariant and flip-invariant.

Just as remarked in §6 for the map $\mathcal{Q} : \partial F_N \to \overline{T} \cup \partial T$, the above map $\mathcal{Q}^1 : L^1(T) \to \overline{T}$ is in general not continuous. The relevance of the set $L^2_{\mathcal{Q}}(T)$ is underlined by the following:

Proposition 8.3. *Let T be an $\mathbb{R}$-tree with a very small action of F_N with dense orbits. Then the subset $L^2_{\mathcal{Q}}(T)$ of $\partial^2 F_N$ is closed, and the map $\mathcal{Q}^2 : L^2_{\mathcal{Q}}(T) \to \overline{T}$ is continuous (for the metric topology on $\overline{T}$).*

Proof. Let $(X_n, Y_n)_{n\in\mathbb{N}}$ be a sequence of points from $L^2_{\mathcal{Q}}(T)$ that converge to $(X, Y) \in \partial^2 F_N$. Let $\mathcal{A}$ be a basis of F_N, P be any point in T, and define $C = 3\mathrm{BBT}(\mathcal{A}, P)$.

Let x_n the largest common prefix of the infinite reduced words in $\mathcal{A}^{\pm 1}$ representing X_n and X, y_n that of Y_n and Y, and h that of X and Y. From the assumption $(X, Y) \in \partial^2 F_N$ we know that X and Y are different, so that h is a finite word: $h \in F(\mathcal{A})$. The assumption $(X_n, Y_n) \overset{n\to\infty}{\longrightarrow} (X, Y)$ implies that the X_n and the Y_n converge to X and Y respectively. Hence, for n big enough one obtains that h is a prefix of both, x_n and y_n. Indeed, since h is the longest common prefix of X and Y, it must also be that of X_n and Y_n. By hypothesis one has $\mathcal{Q}(X_n) = \mathcal{Q}(Y_n)$, so that the BBT property together with Lemma 6.1 ensures that hP lies in a C-neighboorhood of $\mathcal{Q}(X_n) = \mathcal{Q}(Y_n)$. But then, by the definition of $L^1(T)$, the hypothesis $X_n \overset{n\to\infty}{\longrightarrow} X$ and $Y_n \overset{n\to\infty}{\longrightarrow} Y$ implies that X and Y belong to $L^1(T)$. Hence Lemma 6.3 shows that $\mathcal{Q}(X)$ as well as $\mathcal{Q}(Y)$ are contained in a $2C$-neighborhood of hP. Hence passing over to P and $\mathcal{A}$ with arbitrary small $\mathrm{BBT}(\mathcal{A}, P)$ proves that $\mathcal{Q}(X_n) = \mathcal{Q}(Y_n)$ converges to $\mathcal{Q}(X) = \mathcal{Q}(Y)$. □

As a direct consequence of Proposition 8.3 we obtain:

Corollary 8.4. *The set $L^2_{\mathcal{Q}}(T) \subset \partial^2 F_N$ is an algebraic lamination.*

We remark that, contrary to the surface lamination case, the lamination $L^2_{\mathcal{Q}}(T)$ contains all *diagonal leaves*: From the definition it follows directly that, if (X, X') and (X', X'') are in $L^2_{\mathcal{Q}}(T)$ and $X \neq X''$, then (X, X'') is also in $L^2_{\mathcal{Q}}(T)$.

Proposition 8.5. *Let T be an $\mathbb{R}$-tree with a very small F_N-action with dense orbits. Then the lamination $L^2_\Omega(T)$ of Definition 4.1 and the lamination $L^2_{\mathcal{Q}}(T)$ of Definition 8.1 are equal:*

$$L^2_\Omega(T) = L^2_{\mathcal{Q}}(T)$$

Proof. In order to show the inclusion $L^2_{\mathcal{Q}}(T) \subset L^2_\Omega(T)$ it suffices, in view of Theorem 1.1 of [CHL-I], to show that for some basis $\mathcal{A}$ of F_N any word $z = x_1 \ldots x_s$ in the laminary language $\mathcal{L}^{\mathcal{Q}}_{\mathcal{A}}(T)$ associated to the algebraic lamination $L^2_{\mathcal{Q}}(T)$ (via the canonical map $\rho^{\mathcal{A}}_{\mathcal{L}} \rho^2_{\mathcal{A}} : \Lambda^2(F_N) \to \Lambda_{\mathcal{L}}(\mathcal{A})$, see [CHL-I]) is also contained in the laminary language $\mathcal{L}^\Omega_{\mathcal{A}}(T)$ from §4: By extendability of any word in a laminary language we find, for any $k \geq 0$, a "superword" $u_{\mathcal{A}} = x_{-k} \ldots x_1 \ldots x_s \ldots x_{s+k}$ in $\mathcal{L}^{\mathcal{Q}}_{\mathcal{A}}(T)$. Thus, for k sufficiently large, we can use Cooper's cancellation bound and pass to another basis $\mathcal{B}$ with $\mathrm{BBT}(\mathcal{B}, P) < \varepsilon$ and $vol(\mathcal{B}, P) < \varepsilon$, for small $\varepsilon > 0$, such that the word $u_{\mathcal{B}}$ in the basis $\mathcal{B}$, which represents the same element of F_N as $u_{\mathcal{A}}$, contains a subword $v_{\mathcal{B}}$ with the following properties (compare the similar situation considered in Lemma 5.5):

On one hand $v_{\mathcal{B}}$ belongs to the laminary language $\mathcal{L}^{\mathcal{Q}}_{\mathcal{B}}(T)$, which implies by Lemma 3.1 (a) and Lemma 6.1 that $d(P, v_{\mathcal{B}}P) \leq 3\mathrm{BBT}(\mathcal{B}, P) < 3\varepsilon$. As the rank of our free group F_N satisfies $N \geq 2$, we find an element $y \in \mathcal{B}^{\pm 1}$ such that $w_{\mathcal{B}} = v_{\mathcal{B}} y$ is cyclically reduced and satisfies furthermore, by Lemma 3.1 (d), that $\|w_{\mathcal{B}}\|_T \leq d(P, v_{\mathcal{B}}P) + d(P, yP) \leq 3\mathrm{BBT}(\mathcal{B}, P) + \mathrm{vol}(\mathcal{B}, P) < 4\varepsilon$.

On the other hand, by a double application of Cooper's cancellation bound we obtain, since $w_{\mathcal{B}} = v_{\mathcal{B}} y$ is cyclically reduced, for sufficiently large k, that the word $w_{\mathcal{A}}$ in the basis $\mathcal{A}$, which represents the same element of F_N as $w_{\mathcal{B}}$, contains the originally chosen word $z = x_1 \ldots x_s \in \mathcal{L}^{\mathcal{Q}}_{\mathcal{A}}(T)$ as subword in its cyclically reduced part. This shows that $z = x_1 \ldots x_s$ belongs to $\mathcal{L}^{4\varepsilon}_{\mathcal{A}}(T)$, and hence, as $\varepsilon > 0$ was picked arbitrarily small, to $\mathcal{L}^\Omega_{\mathcal{A}}(T)$.

For the converse inclusion $L^2_\Omega(T) \subset L^2_{\mathcal{Q}}(T)$ we consider any $(X, X') \in L^2_\Omega(T)$ and apply Proposition 5.3 to deduce that both, X and X' belong to $L^1(T)$. Hence there are well defined points $R = \mathcal{Q}(X)$ and $Q = \mathcal{Q}(X')$ in $\overline{T}$. If $R \neq Q$, then for any small $\varepsilon > 0$ we can pass to a basis $\mathcal{B}$ of F_N with $12\mathrm{BBT}(\mathcal{B}, P) < 4\varepsilon < d(R, Q)$, and we consider the biinfinite word $Z_{\mathcal{B}} = \ldots y_{i-1} y_i y_{i+1} \ldots = \rho_{\mathcal{B}}(X, X')$ in this basis. Any subword $w = y_{-k} \ldots y_l$ of $Z_{\mathcal{B}}$ with k and l sufficiently large satisfies $d(y_1 \ldots y_l P, R) < 3\mathrm{BBT}(\mathcal{B}, P) < \varepsilon$, as well as $d((y_{-k} \ldots y_0)^{-1} P, Q) < 3\mathrm{BBT}(\mathcal{B}, P) < \varepsilon$, as follows from Lemma 6.3.

As $d(P, wP) = d(y_1 \ldots y_l P, (y_{-k} \ldots y_0)^{-1} P)$, this gives $4\varepsilon < d(R, Q) \leq d(y_1 \ldots y_l P, R) + d(P, wP) + d((y_{-k} \ldots y_0)^{-1} P, Q) < d(P, wP) + 2\varepsilon$, and thus $d(P, wP) > 2\varepsilon$. Hence, if w is subword of any cyclically reduced word W in $\mathcal{B}$, we obtain from Lemma 3.1 (c) the inequalities $\|W\|_T \geq d(P, wP) - 2\mathrm{BBT}(\mathcal{B}, P) \geq \varepsilon$. This shows that w does not belong to $\mathcal{L}^\varepsilon_{\mathcal{B}}(T)$, for $\varepsilon > 0$ small, and hence not to $\mathcal{L}_{\mathcal{B}}(T)$, contradicting the assumption $Z_{\mathcal{B}} \in \rho^2_{\mathcal{B}}(L^2_\Omega(T))$. Thus we have proved that $R = Q$, which shows that (X, X') belongs to $L_{\mathcal{Q}}(T)$. □

9 From the boundary of Outer space to the space of laminations

The group $\mathrm{Out}(F_N)$ acts canonically (from the left) on the space $\Lambda^2(F_N)$ of algebraic laminations (see [CHL-I], §8), but it also acts (from the right!) on the space cv_N and on its "Thurston boundary" ∂cv_N, and this induces an action on $CV_N \cup \partial CV_N$ (see §2). This right action is defined as follows: For any $\alpha \in \mathrm{Aut}(F_N)$ and any tree $T \in \overline{\mathrm{cv}}_N$, the length function of the image tree $T\alpha_*$ is given by

$$\|w\|_{T\alpha_*} = \|\alpha(w)\|_T \quad \text{for every} \quad w \in F_N \,.$$

Proposition 9.1. *The map*

$$\lambda^2 : \partial CV_N \to \Lambda^2(F_N), \; [T] \mapsto L^2(T)$$

is Out(F_N)-anti-equivariant: For any automorphism α of F_N and any $[T] \in \partial CV_N$ one has $\alpha^{-1}(L^2(T)) = L^2(T\alpha_*)$.

This follows directly from the definition of $L^2(T) = L^2_\Omega(T) = \bigcap\limits_{\varepsilon > 0} L^2_\varepsilon(T)$ in §4, since we observe, for every $\varepsilon > 0$, that

$$\alpha^{-1}(\Omega_\varepsilon(T)) = \Omega_\varepsilon(T\alpha_*)$$

and hence

$$\alpha^{-1}(L^2_\varepsilon(T)) = L^2_\varepsilon(T\alpha_*) \,.$$

□

However, it is important to point out that the map λ^2 is not continuous. For example, consider the case $F_3 = F(\{a, b, c\})$, and let D be the *Dehn twist* automorphism given by $a \mapsto bab^{-1}, b \mapsto b, c \mapsto c$. Then in the last section of [CL95] it is shown that every tree $T \in cv_3 \cup \partial cv_3$ with $\|b\|_T > 0$ converges projectively under iteration of D to the simplicial tree $\widetilde{\Gamma}_b$ which is the Bass-Serre tree of the graph of groups decomposition $F(a,b,c) = < a, b > *_{<b=b'>} < b', c >$. Thus, if T is, for example, the simplicial tree obtained from the rose $\mathcal{R}_{\{a,b,c\}}$ by contracting in the universal covering the edge labelled c equivariantly (while leaving all other edges of same length), one gets that $L^2(T) = L^2(c)$, compare Remark 4.3. Replacing a by $b^m ab^{-m}$ defines a family of new trees $T_m \in \partial cv_3$, which projectivize to points $[T_m] \in \partial CV_3$ that give precisely the D-orbit of $[T] \in \partial CV_3$, which has $[\widetilde{\Gamma}_b]$ as forward and backward limit point. On the other hand, we obtain from Remark 4.3 that $L^2(T_m) = L^2(c)$ for all $m \in \mathbb{Z}$. Since $L^2(\widetilde{\Gamma}_b)$ contains $L^2(c)$, but is much larger (for example $L^2(a)$ is equally contained in $L^2(\widetilde{\Gamma}_b)$), this example illustrates that there exist sequences of trees T_k converging to T such that $L^2(T_k)$ converges into $L^2(T)$, but not to $L^2(T)$.

Remark 9.2. The above example is in fact typical in that the dual lamination of the limit tree contains but is in general bigger than the limit of the dual laminations, for a convergent sequence of $\mathbb{R}$-trees from $\overline{\mathrm{CV}}_N$. More details and a precise statement of this fact is given in [CHL3].

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